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COMMENT

Permutation operators in Hilbert space gained via IWOP technique

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Abstract. A kind of permutation operators in Hilbert space are derived in terms of the technique of integration within ordered product (IWOP). These operators are shown to be quantum maps imaged by certain permutation transformations in classical space. Some operator identities about the relationship between permutation operators and transposition operators are also derived.

1. Introduction

As is well known, permutation transformation is an important topic both in group theory [1] and in quantum mechanics when exchange symmetry is considered [2]. However, the problem of what are n-body permutation operators in Hilbert space has not received enough attention in the literature. The purpose of this comment is to study these operators by directly using the newly developed IWOP (integration within ordered product) technique. This technique was introduced into quantum mechanics in [3] and has provided us with a new approach to studying a variety of problems in quantum optics [4–6] and in the quantum-classical transition regime [7–9]. In this work, we want to obtain: (i) some new expressions of permutation operators which are quantum maps imaged by certain permutation transformations in classical space and (ii) some new operator identities regarding the relationship of permutation operators. In § 2, we deal with the transposition operator since any permutation is equivalent to a finite number of transpositions. In § 3 we derive three-body permutation operators; the method we use in this section can be easily generalised to derive n-body permutation operators, as discussed in § 4.

2. Transposition operator gained via IWOP technique

Consider a transposition operator p_{21} whose action on the two-mode coordinate eigenstate $|q_1 q_2\rangle$ obeys

$$p_{21}|q_1 q_2\rangle = |q_2 q_1\rangle \tag{2.1}$$

where $|q_1 q_2\rangle$ is the tensor product of $|q_1\rangle$ and $|q_2\rangle$, e.g.

$$|q_1 q_2\rangle = \pi^{-1/2} \exp[-\tfrac{1}{2}(q_1^2 + q_2^2) + \sqrt{2}(q_1 a_1^\dagger + q_2 a_2^\dagger) - \tfrac{1}{2}(a_1^{\dagger 2} \dagger a_2^{\dagger 2})]|00\rangle \tag{2.2}$$

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and

$$|q_2q_1\rangle = \pi^{-1/2}\exp[-\tfrac{1}{2}(q_1^2+q_2^2)+\sqrt{2}(q_2a_1^\dagger+q_1a_2^\dagger)-\tfrac{1}{2}(a_1^{\dagger 2}+a_2^{\dagger 2})]|00\rangle. \tag{2.3}$$

Here $a_k^\dagger$ $(k=1,2)$ are Bose creation operators, satisfying $[a_k, a_{k'}^\dagger]=\delta_{kk'}$, $|00\rangle$ is the ground state of the two-mode harmonic oscillator, the projection operator $|00\rangle\langle 00|$ is

$$|00\rangle\langle 00| = :\exp(-a_1^\dagger a_1 - a_2^\dagger a_2): \tag{2.4}$$

where $:\ldots:$ denotes the normal product. Using the completeness relation of $|q_1q_2\rangle$

$$\iint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,|q_1q_2\rangle\langle q_1q_2| = 1 \tag{2.5}$$

we can easily obtain the coordinate representation of p_{21}

$$p_{21} = \iint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,|q_2q_1\rangle\langle q_1q_2|. \tag{2.6}$$

Using (2.3), (2.4) and the IWOP technique, we perform the integration of (2.6) and get

$$\begin{aligned} p_{21} = \frac{1}{\pi}\iint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\, &:\exp[-q_1^2-q_2^2+\sqrt{2}(q_2a_1^\dagger+q_1a_2^\dagger+q_1a_1+q_2a_2) \\ &-\tfrac{1}{2}(a_1^\dagger+a_1)^2-\tfrac{1}{2}(a_2^\dagger+a_2)^2]: \\ &= :\exp(a_2^\dagger a_1 + a_1^\dagger a_2 - a_1^\dagger a_2 - a_2^\dagger a_2):. \end{aligned} \tag{2.7}$$

Further, by virtue of the operator identity

$$:\exp[\lambda(a_2^\dagger-a_1^\dagger)(a_1-a_2)]: = \exp[-\tfrac{1}{2}\ln(1-2\lambda)(a_2^\dagger-a_1^\dagger)(a_1-a_2)] \tag{2.8}$$

p_{21} can be put into

$$p_{21} = \exp[-\tfrac{1}{2}\mathrm{i}\pi(a_2^\dagger-a_1^\dagger)(a_1-a_2)] \tag{2.9}$$

or

$$p_{21} = \exp(\tfrac{1}{2}\mathrm{i}\pi J_y)\exp(\mathrm{i}\pi a_1^\dagger a_1)\exp(-\tfrac{1}{2}\mathrm{i}\pi J_y) \tag{2.10}$$

where J_y is introduced by $J_y=(1/2\mathrm{i})(a_1^\dagger a_2 - a_2^\dagger a_1)$. Since $\exp[\mathrm{i}\pi a_1^\dagger a_1| = (-1)^{N_1}$, $N_1 = a_1^\dagger a_1$, it is easy to prove that p_{21} is its own inverse, e.g.

$$p_{21}^2 = \exp(\tfrac{1}{2}\mathrm{i}\pi J_y)(-1)^{N_1}(-1)^{N_1}\exp(-\tfrac{1}{2}\mathrm{i}\pi J_y) = 1.$$

From (2.6), one can easily see that p_{21} is Hermitian

$$p_{21}^\dagger = \iint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,|q_1q_2\rangle\langle q_2q_1| = p_{21}. \tag{2.11}$$

Moreover, according to (2.9), p_{21} is also unitary $p_{21}^\dagger = p_{21}^{-1}$. As a consequence of (2.10), we get

$$p_{21}a_1^\dagger p_{21} = a_2^\dagger \qquad p_{21}a_2^\dagger p_{21} = a_1^\dagger \tag{2.12}$$

which confirms that p_{21} is the transposition operator in Hilbert space, and the coordinate representation (2.6) of p_{21} shows that p_{21} is the quantum map imaged by interchanging $q_1 \leftrightarrow q_2$ in classical space.

3. Three-body permutation operators

Consider, therefore, the three-mode coordinate eigenstate

$$|q_1q_2q_3\rangle = \pi^{-3/4}\exp[-\tfrac{1}{2}(q_1^2+q_2^2+q_3^2) + \sqrt{2}(q_1a_1^\dagger+q_2a_2^\dagger+q_3a_3^\dagger)-\tfrac{1}{2}(a_1^{\dagger 2}+a_2^{\dagger 2}+a_3^{\dagger 2})]|000\rangle \tag{3.1}$$

where $|000\rangle$ is the three-mode vacuum state, satisfying

$$|000\rangle\langle 000| = :\exp[-a_1^\dagger a_1 - a_2^\dagger a_2 - a_3^\dagger a_3]:. \tag{3.2}$$

In this case, there exist six permutation transformations, denoted by (rst), where r, s, t is an arbitrary permutation of the numbers 1, 2, 3. The action of (rst) on $\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ obeys

$$\left|(rst)\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right\rangle = \left|\begin{pmatrix} q_r \\ q_s \\ q_t \end{pmatrix}\right\rangle = \pi^{-3/4}\exp[-\tfrac{1}{2}(q_r^2+q_s^2+q_t^2)+\sqrt{2}(q_ra_1^\dagger+q_sa_2^\dagger+q_ta_3^\dagger) -\tfrac{1}{2}(a_1^{\dagger 2}+a_2^{\dagger 2}+a_3^{\dagger 2})]|000\rangle. \tag{3.3}$$

The six permutation transformations can have the following representation

$$(231)=\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix} \qquad (312)=\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix} \qquad (213)=\begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$$
$$(132)=\begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix} \qquad (321)=\begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix} \qquad (123)=\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}\equiv \mathbb{1}. \tag{3.4}$$

It is easy to prove that these matrices constitute a group. The multiplication table of the group can be easily obtained by the matrix product†, for example,

$$(132)(213)=(231) \qquad (231)(312)=\mathbb{1}. \tag{3.5}$$

† It must be pointed out that the multiplication table derived from (3.4) is different from that in [1] where a permutation is characterised by an array

$$\begin{pmatrix} 1 & 2 & 3 \\ n_1 & n_2 & n_3 \end{pmatrix} \equiv T_{n_1n_2n_3}$$

which indicates the operation of putting the objects in slot i into slot n_i, and the multiplication rule can be shown through the following example:

$$T_{132}T_{213}=\begin{pmatrix} 1&2&3\\1&3&2 \end{pmatrix}\begin{pmatrix} 1&2&3\\2&1&3 \end{pmatrix}=\begin{pmatrix} 2&1&3\\3&1&2 \end{pmatrix}\begin{pmatrix} 1&2&3\\2&1&3 \end{pmatrix}=\begin{pmatrix} 1&2&3\\3&1&2 \end{pmatrix}.$$

In deducing this result the columns of T_{132} were arranged so that the top row of T_{132} was identical with the bottom row of T_{213}. This manipulation does not match with the matrix product of (3.5). In [2], the product rule of permutation operators is defined in a similar manner to [1].

Now consider the operator

$$p_{231} \equiv \iiint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3 \left|(231)\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right|$$

$$= \iiint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3 \left|\begin{pmatrix} q_2 \\ q_3 \\ q_1 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right| \tag{3.6}$$

whose Hermite conjugate operator is

$$p_{231}^{\dagger} = \iiint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3 \left|\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_2 \\ q_3 \\ q_1 \end{pmatrix}\right|$$

$$= \iiint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3 \left|\begin{pmatrix} q_3 \\ q_1 \\ q_2 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right| = p_{312}. \tag{3.7}$$

Using (3.2) and (3.3), and the IWOP technique, we can carry out the integration in (3.6) and get

$$\begin{aligned} p_{231} = \pi^{-3/2} \iiint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3\ {:}\exp[&-(q_1^2+q_2^2+q_3^2) \\ &+\sqrt{2}(q_2 a_1^{\dagger}+q_3 a_2^{\dagger}+q_1 a_3^{\dagger}+q_1 a_1+q_2 a_2+q_3 a_3) \\ &-\tfrac{1}{2}(a_1+a_1^{\dagger})^2-\tfrac{1}{2}(a_2+a_2^{\dagger})^2-\tfrac{1}{2}(a_3+a_3^{\dagger})^2]{:} \\ = {:}\exp&\left[(a_1^{\dagger}a_2^{\dagger}a_3^{\dagger})[(231)-\mathbb{1}]\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right]{:}. \end{aligned} \tag{3.8}$$

Comparing (3.6) with (3.8), we see that p_{231} is the quantum map imaged by the permutation transformation (231) in q_1, q_2, q_3 space. A trivial generalisation of (3.8) is

$$p_{rst} \equiv \iiint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3 \left|(rst)\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right|$$

$$= {:}\exp\left[(a_1^{\dagger}a_2^{\dagger}a_3^{\dagger})[(rst)-\mathbb{1}]\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right]{:}. \tag{3.9}$$

Since $|\det(rst)| = 1$, it is easy to see that

$$p_{rst}p_{rst}^{\dagger} = \iiint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3 \left|(rst)\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right\rangle\left\langle(rst)\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right| = 1$$

which implies that $p_{rst}^{\dagger}=p_{rst}^{-1}$, e.g. every permutation operator is unitary. Further, in terms of (3.6) we can show that p_{231} can be decomposed as the product of transposition operators, e.g.

$$p_{231}=\iiint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3\,\mathrm{d}q_1'\,\mathrm{d}q_2'\,\mathrm{d}q_3' \left|\begin{pmatrix} q_1\\ q_3\\ q_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1'\\ q_2'\\ q_3'\end{pmatrix}\right| \delta(q_1-q_2')\delta(q_2-q_1')\delta(q_3-q_3')$$

$$=\iiint_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3 \left|\begin{pmatrix} q_1\\ q_3\\ q_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1\\ q_2\\ q_3\end{pmatrix}\right| \iiint_{-\infty}^{\infty} \mathrm{d}q_1'\,\mathrm{d}q_2'\,\mathrm{d}q_3' \left|\begin{pmatrix} q_2'\\ q_1'\\ q_3'\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1'\\ q_2'\\ q_3'\end{pmatrix}\right|$$

$$=p_{132}p_{213}. \tag{3.10}$$

Similarly, we have

$$p_{312}=p_{213}p_{132}. \tag{3.11}$$

By introducing the three-mode coherent state

$$|z_1z_2z_3\rangle=\exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2+|z_3|^2)+z_1a_1^{\dagger}+z_2a_2^{\dagger}+z_3a_3^{\dagger}]|000\rangle \tag{3.12}$$

and using the property

$$:f(a_i,a_i^{\dagger}):|z_1z_2z_3\rangle=f(z_i,a_i^{\dagger})|z_1z_2z_3\rangle \tag{3.13}$$

we can immediately obtain the effect of (3.8) acting on $|z_1z_2z_3\rangle$

$$p_{231}|z_1z_2z_3\rangle=|z_2z_3z_1\rangle. \tag{3.14}$$

In order to further analyse p_{rst}, we appeal to the following operator identity [10] (see also the appendix of this present paper):

$$\exp\left[\sum_{i,j} a_i^{\dagger}\Lambda_{ij}a_j\right]=:\exp\left[\sum_{i,j} a_i^{\dagger}(\mathrm{e}^{\Lambda}-\mathbb{1})_{ij}a_j\right]:. \tag{3.15}$$

With the help of (3.15), we can reform, for example, p_{312} into the following form, which seems to be new:

$$p_{312}=\exp\left[(a_1^{\dagger}a_2^{\dagger}a_3^{\dagger})\ln\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\right] \tag{3.16}$$

where the logarithm can be easily evaluated because the matrix in (3.16) can be diagonised. The result is

$$\begin{gathered}\chi\equiv\ln\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}=\begin{pmatrix}\gamma&\alpha&\beta\\\beta&\gamma&\alpha\\\alpha&\beta&\gamma\end{pmatrix}\\ \alpha\equiv\frac{1-\sqrt{3}\mathrm{i}}{3\sqrt{3}}\pi \qquad \beta\equiv-\frac{1+\sqrt{3}\mathrm{i}}{3\sqrt{3}}\pi \qquad \gamma\equiv\frac{2\pi\mathrm{i}}{3}.\end{gathered} \tag{3.17}$$

Since $\alpha^* = -\beta$, $\gamma^* = -\gamma$, $e^{-\chi} = (231)$, one can easily confirm that $p_{312}^{\dagger} = p_{312}^{-1} = p_{231}$. As a result of (2.9), (3.11) and (3.16), the breakdown of p_{312} into the product of p_{213} and p_{132} can be rewritten as a new operator identity

$$\exp\left[(a_1^\dagger a_2^\dagger a_3^\dagger)\chi\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\right] = \exp[-\tfrac{1}{2}\mathrm{i}\pi(a_2^\dagger - a_1^\dagger)(a_1 - a_2)]\exp[-\tfrac{1}{2}\mathrm{i}\pi(a_3^\dagger - a_2^\dagger)(a_2 - a_3)]. \tag{3.18}$$

Further, by virtue of (A2) in the appendix, we obtain the permutation properties in Hilbert space

$$p_{312}a_1^\dagger p_{312}^{-1} = a_2^\dagger \qquad p_{312}a_2^\dagger p_{312}^{-1} = a_3^\dagger \qquad p_{312}a_3^\dagger p_{312}^{-1} = a_1^\dagger. \tag{3.19}$$

The above discussions can be generalised to derive n-body permutation operators in Hilbert space.

4. N-body permutation operators

Let $(uv\ldots w)$ denote an n-body permutation matrix

$$(uv\ldots w) = \begin{pmatrix}\delta_{u1} & \delta_{u2} & \ldots & \delta_{un}\\ \delta_{v1} & \delta_{v2} & \ldots & \delta_{vn}\\ \vdots & & & \\ \delta_{w1} & \delta_{w2} & \ldots & \delta_{wn}\end{pmatrix} \tag{4.1}$$

where $u, v, \ldots, w$ is an arbitrary permutation of the numbers $1, 2, \ldots, n$. There exist $n!$ permutation matrices which constitute, in the sense of matrix product, a group. After introducing the n-mode coordinate eigenstate

$$\left|\begin{pmatrix}q_1\\q_2\\\vdots\\q_n\end{pmatrix}\right\rangle = \pi^{-n/4}\exp[-\tfrac{1}{2}(q_1^2 + q_2^2 + \ldots + q_n^2) + \sqrt{2}(q_1 a_1^\dagger + q_2 a_2^\dagger + \ldots + q_n a_n^\dagger)$$
$$-\tfrac{1}{2}(a_1^{\dagger 2} + a_2^{\dagger 2} + \ldots + a_n^{\dagger 2})]|00\ldots 0\rangle \tag{4.2}$$

we can construct the permutation operator $p_{uv\ldots w}$ by

$$p_{uv\ldots w} = \int_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\ldots\mathrm{d}q_n \left|(uv\ldots w)\begin{pmatrix}q_1\\q_2\\\vdots\\q_n\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1\\q_2\\\vdots\\q_n\end{pmatrix}\right|$$
$$= \int_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2\ldots\mathrm{d}q_n \left|\begin{pmatrix}q_u\\q_v\\\vdots\\q_w\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1\\q_2\\\vdots\\q_n\end{pmatrix}\right|. \tag{4.3}$$

Again using the IWOP technique, we can put $p_{uv\dots w}$ into the normal product form

$$p_{uv\dots w}=:\exp\left[(a_1^\dagger a_2^\dagger\dots a_n^\dagger)[(uv\dots w)-\mathbb{1}]\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}\right]: \tag{4.4}$$

where $\mathbb{1}$ is the $n\times n$ unit matrix. As a consequence of (A4), equation (4.4) becomes

$$p_{uv\dots w}=\exp\left[(a_1^\dagger a_2^\dagger\dots a_n^\dagger)\ln(uv\dots w)\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}\right]. \tag{4.5}$$

Further, consider the product $p_{uv\dots w}$ and $p_{u'v'\dots w'}$; using (4.3) we have

$$p_{uv\dots w}p_{u'v'\dots w'}=\int_{-\infty}^{\infty}\mathrm{d}q_1\,\mathrm{d}q_2\dots\mathrm{d}q_n\left|(uv\dots w)(u'v'\dots w')\begin{pmatrix}q_1\\q_2\\\vdots\\q_n\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1\\q_2\\\vdots\\q_n\end{pmatrix}\right| \tag{4.6}$$

which yields

$$\exp\left[(a_1^\dagger a_2^\dagger\dots a_n^\dagger)\ln(uv\dots w)\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}\right]\exp\left[(a_1^\dagger a_2^\dagger\dots a_n^\dagger)\ln(u'v'\dots w')\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}\right]$$

$$=\exp\left[(a_1^\dagger a_2^\dagger\dots a_n^\dagger)\ln[(uv\dots w)(u'v'\dots w')]\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}\right]. \tag{4.7}$$

In summary, by means of the IWOP technique, we have found a new direct approach to obtaining both the normally ordered and explicit permutation operator in Hilbert space.

Appendix

Let Λ be a 3×3 matrix. From the operator identity

$$\mathrm{e}^{A}B\,\mathrm{e}^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\dots \tag{A1}$$

and the commutation relation $[a_i,a_j^\dagger]=\delta_{ij}$, we can obtain

$$\exp\Big(\sum_{i,j}a_i^\dagger\Lambda_{ij}a_j\Big)a_l^\dagger\exp\Big(-\sum_{i,j}a_i^\dagger\Lambda_{ij}a_j\Big)=\sum_i a_i^\dagger(\mathrm{e}^{\Lambda})_{il}. \tag{A2}$$

Using the overcompleteness relation of the coherent state

$$\int \prod_{i=1}^{3} \frac{d^2 z_i}{\pi} |z_1 z_2 z_3\rangle\langle z_1 z_2 z_3|$$

$$= \int \prod_{i=1}^{3} \frac{d^2 z_i}{\pi} :\exp\left(\sum_i (-|z_i|^2 + z_i a_i^\dagger + z_i^* a_i - a_i^\dagger a_i)\right): = 1 \tag{A3}$$

and (A2), we have

$$\exp\left(\sum_{i,j} a_i^\dagger \Lambda_{ij} a_j\right)$$

$$= \int \prod_{i=1}^{3} \frac{d^2 z_i}{\pi} \exp\left(\sum_{i,j} a_i^\dagger \Lambda_{ij} a_j\right) \exp\left(\sum_i z_i a_i^\dagger\right)$$

$$\times \exp\left(-\sum_{i,j} a_i^\dagger \Lambda_{ij} a_j\right) |000\rangle\langle z_1 z_2 z_3| \exp\left(-\sum_i \frac{|z_i|^2}{2}\right)$$

$$= \int \prod_{i=1}^{3} \frac{d^2 z_i}{\pi} :\exp\left(\sum_i [-|z_i|^2 + \sum_l a_i^\dagger (e^\Lambda)_{il} z_l + z_i^* a_i - a_i^\dagger a_i]\right):$$

$$= :\exp\left(\sum_{i,j} a_i^\dagger (e^\Lambda - \mathbb{1})_{ij} a_j\right): \tag{A4}$$

which can be generalised to the case of Λ being an $n \times n$ matrix.

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